Goals:

1. Finish CEP $\Rightarrow$ QWEP
2. Start Complexity Theory

Last time:

Thm (Kirchberg) $(C^*(\mathbb{F}_\infty), \mathcal{B}(H))$ is a nuclear pair:

$$C^*(\mathbb{F}_\infty) \otimes_{\min} \mathcal{B}(H) = C^*(\mathbb{F}_\infty) \otimes_{\max} \mathcal{B}(H).$$

Exercise: If $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A} \otimes_{\min} \mathcal{C} \subseteq \mathcal{B} \otimes_{\min} \mathcal{C}$ for any $\mathcal{C}$.

Issue: In general not true for $\otimes_{\max}$.
$\mathcal{A} \otimes_{\max} \mathcal{C} \to \mathcal{B} \otimes_{\max} \mathcal{C}$ need not be isometric.

One instance when it does hold:

$A \subseteq A^{**}$ ← ?

Universal representation $\pi_u : A \to \mathcal{B}(\mathcal{H}_u)$

A^{**} = von Neumann subalg. of $\mathcal{B}(\mathcal{H}_u)$ generated $\pi_u(A)$.

$$A \otimes_{max} C \subseteq A^{**} \otimes_{max} C.$$

weak cond. expectation

Fact: If $A \subseteq \mathcal{B}$ and there is a ucp $\mathcal{B} \to A^{**}$ that restricts to the identity on A, then

$$A \otimes_{max} C \subseteq \mathcal{B} \otimes_{max} C.$$

Thm For $A \subseteq \mathcal{B}$, TFAE:

(1) $(\forall C)$ $A \otimes_{max} C \subseteq \mathcal{B} \otimes_{max} C$ ←

② There is a weak cond exp $\mathcal{B} \to A^{**}$.
weakly cp-complemented

When A is weakly cp-comp in $\mathcal{B}(H)$, say A has the weak expectation property (WEP).

Thm (Kirchberg) A has WEP iff $(A, C^*(\mathbb{F}_\infty))$ nuclear pair.

Q Does $C^*(\mathbb{F}_\infty)$ have WEP?
I.e. Is $(C^*(\mathbb{F}_\infty), C^*(\mathbb{F}_\infty))$ a nuclear pair?

Tautology: If $C^*(\mathbb{F}_\infty)$ has WEP, then every separable C^*-algebra has QWEP, that is, is a quotient of something with WEP.

Converse is true: If $C^*(\mathbb{F}_\infty)$ has QWEP, then it actually has WEP.

Reason: $C^*(\mathbb{F}_\infty)$ has lifting property (LP):

$$\begin{tikzcd} & \mathcal{B} \arrow[d] \\ C^*(\mathbb{F}_\infty) \arrow[ur, dashed, "\exists \text{ ucp}"] \arrow[r, "\text{hom}"'] & \mathcal{B}/\mathcal{J} \end{tikzcd}$$

LLP + QWEP $\Rightarrow$ WEP

L = local

Thm (Kirchberg) TFAE:

Kirchberg's QWEP Problem

① $C^*(\mathbb{F}_\infty)$ has WEP

② $(C^*(\mathbb{F}_k), C^*(\mathbb{F}_k))$ nuclear pair for some (equiv any) $k \in \{2,3,\dots\} \cup \{\infty\}$.

③ Every sep C^*-alg has QWEP
④ LLP $\Rightarrow$ WEP.

Now: CEP $\Rightarrow$ QWEP

(i) $\mathcal{R}$ has WEP ($\mathcal{R}$ is "injective")
(ii) $\therefore$ $\ell^\infty(\mathcal{R})$ has WEP
(iii) $\therefore$ $\mathcal{R}^u$ has QWEP: $\mathcal{R}^u = \ell^\infty(\mathcal{R})/c_u$
(iv) $\therefore$ If M is a tracial vNa and $M \subseteq \mathcal{R}^u$, then M is weakly cp-comp in $\mathcal{R}^u$ and so M has QWEP.

Moral: CEP $\Rightarrow$ <u>all tracial vNas have QWEP.</u>

Since A is weakly cp-comp in the vNa A^{**}, to show A has QWEP, Enough to show A^{**} has QWEP.

Issue: $\boxed{A^{**}}$ probably doesn't have a trace.

(v) For a general vNa M, there is a 1-parameter subgroup σ_t of automorphisms of M (modular group) so that $M \rtimes_{\sigma_t} \mathbb{R}$ is semifinite. (Takesaki)

Finite vNas have QWEP $\Rightarrow$ semifinite ones.

(vi) M is weakly cp-comp in $M \rtimes_{\sigma_t} \mathbb{R}$.

Complexity Theory

Turing machine

If $\underline{M}$ is a Turing machine, let $f^{\underline{M}}: \{0,1\}^* \to \{0,1\}$ denote the partial function it computes.

$\{0,1\}^*$ = set of finite binary sequences

If $T: \mathbb{N} \to \mathbb{N}$ is a function, say $\underline{M}$ runs in time $T(n)$ if: for every $z \in \{0,1\}^*$, upon input z, the machine halts in $\leq T(|z|)$ many steps.

$\underline{M}$ runs in polynomial time if...

exponential time $\sim 2^{|z|}$

doubly exponential time

language $L \subseteq \{0,1\}^*$

"codes for problem instances"

complexity class set of languages

Least complex P = set of languages L so that there is a poly time Turing machine that computes χ_L = char. function of L

EXP = same but w/ exp time machine

$P \subsetneq EXP$

↑ Time Hierarchy Thm

NP = set of languages L for which there is a poly time machine $\underline{M}$ and poly. $p(n)$ so that: "witness"

- if $z \in L$, then there is $w \in \{0,1\}^{p(|z|)}$ so that $\underline{M}(z,w) = 1$.
- If $z \notin L$, then for every $w \in \{0,1\}^{p(|z|)}$, $M(z,w) = 0$

Verifier & prover

Input: z

Prover: Trying to convince verifier $z \in L$.
w is her proof ("All powerful")

Verifier: Checks if w is a valid proof that $z \in L$.

using $\underline{M}$

example Graph isomorphism

$L = \{(G_1, G_2) : G_1, G_2$ finite graphs, $G_1 \cong G_2\}$

Belongs to NP

N: nondeterministic

$P \subseteq NP \subseteq EXP$

$P = NP$?

Also NEXP, NEEXP...

$NP \subsetneq NEXP \subsetneq NEEXP$

PSPACE = set of languages L that can be decided using machines that use a poly. amount of space.

$P \subseteq PSPACE$

$NP \subseteq \underline{PSPACE} \subseteq EXP \subseteq \underline{NEXP}$

$\overset{\uparrow}{=?}$ PSPACE = NEXP?

BPP: Just like NP except for a randomly chosen $r \in \{0,1\}^{p(|z|)}$,

$Prob(\underline{M}(z,r) = \chi_L(z)) \geq \frac{2}{3}$.

Interactive proofs

$$\underline{V}(z) = a_1$$
$$P(z, a_1) = a_2$$
$$\underline{V}(z, a_1, a_2) = a_3$$
$$P(z, a_1, a_2, a_3) = a_4$$
$$\vdots$$

} k rounds

Then $V(z, a_1, \ldots, a_{2k}) =$ yes or no

This is just NP in disguise.

$\underline{V}$ poly machine, $r \in \{0,1\}^{p(|z|)}$

$$\underline{V}(z, r) = a_1$$
$$P(z, a_1) = a_2$$
$$\underline{V}(z, r, a_1, a_2) = a_3$$
$$P(z, a_1, a_2, a_3) = a_4$$
$$\vdots$$

} k rounds

$\underline{V}(z, r, a_1, \ldots, a_{2k}) =$ yes or no

L belongs to IP if there is such $\underline{V}$ and p so that:

- If $z \in L$, then there is a function P so that $\text{Prob}(\underline{V}(z, r, a_1, \ldots, a_{2k}) = 1) \geq \frac{2}{3}$
- If $z \notin L$, then for every such P, $\text{Prob}(\underline{V}(z, r, a_1, \ldots, a_{2k}) = 1) \leq \frac{1}{3}$.

<u>example</u> Graph non-isom. is in IP.

Protocol: Input (G_1, G_2).

Verifier flips a coin, getting $i \in \{1, 2\}$.
Randomly picks a permutation σ of vertices of G_i, obtaining $H \cong G_i$

question

Prover returns $a \in \{1, 2\}$.

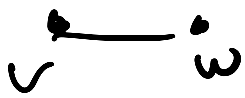
If $G_1 \not\cong G_2$, prover can always get it right.

If $G_1 \cong G_2$, prover can do no better than guessing.

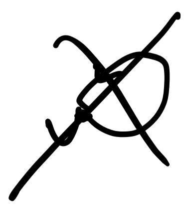
Graph: $G = (V, E)$ $\quad V$ vertex set

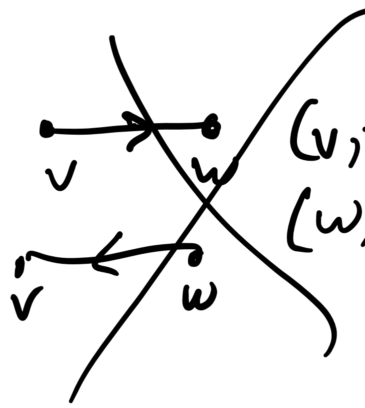
$E \subseteq V \times V$

$(v, w) \in E$



$(v, v) \notin E$





$(v, w) \in E$

$(w, v) \notin E$